JOURNAL OF NUMBER THEORY 33, 182–198 (1989)

The Hausdorff Dimensions of Some Continued Fraction Cantor Sets

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Communicated by P. T. Bateman

Received November 17, 1987; revised August 4, 1988

We give a new method for finding the Hausdorff dimension of the sets E_n consisting of the real numbers which have all their continued fraction partial quotients $\leqslant n$. In particular, we show that dim $E_2 \in (.53128049, .53128051)$.

1. INTRODUCTION

The Hausdorff, or fractional, dimension of a set E is, loosely speaking, the limiting value of the ratio $\log N/\log M$, if $N = N(M)$ is the number of open sets of radius $1/M$ required to cover E. For a full definition see [1].

The Cantor "middle third" set has dimension log 2/log 3, for instance. If instead of putting conditions on the base 3 digits of a number, we require that it have small continued fraction partial quotients ($\leqslant n$), we get the sets E_n of the abstract: $E_n := \{x \in \mathbf{R}\colon \text{if } x = [0; a_1, a_2, a_3, \ldots] \text{ then for all } i \geqslant 1, 1 \leqslant a_i \leqslant n\}$. Let $V_n(r) := \{v = (v_1, v_2, \ldots, v_r)\colon 1 \leqslant v_i \leqslant n \text{ for } 1 \leqslant i \leqslant r\}$, and for $v \in V_n(r)$, let $[v] = [v_1, v_2, \ldots, v_r] = 1/(v_1 + 1/(v_2 + \cdots + 1/v_r)\ldots)$, and let $\langle v \rangle$ be the denominator of $[v]$. In 1941, Good [5] studied these sets. In 1977, Cusick proved that is s is the exponent of convergence for the series

$$\sum_{r=1}^{\infty} \sum_{v \in B_n(r)} \langle v \rangle^{-s}, \tag{1.1}$$

then the Hausdorff dimension of E_n is $\frac{1}{2}s$ [2]. In a sense this solves the problem, since it is not hard to arrive at a procedure which, given time, determines whether or not the series above converges. Unfortunately, the most obvious procedures are far too slow, and various improvements have been advanced, especially for E_2, which is now know to have Hausdorff dimension between 0.5312 and 0.5314 [1].

0022-314X/89 $3.00

In two recent papers [6, 7], the series $\sum \langle v \rangle^{-s}$ was studied for different reasons, and a tentative estimate of 1.06256 given for the exponent of convergence. This series has also come up in [2, 3, 4]. I thank Prof. David Boyd of the University of British Columbia for pointing out to me the connection to Hausdorff dimension, and providing several valuable references.

The new ingredient in our method is that we look at the functions

$$\phi^r(s, t) := \sum_{v \in V_n(r)} \langle v \rangle^{-s}(1 + [v]t)^{-s}, \qquad 0 \leqslant t \leqslant 1, \tag{1.2}$$

instead of just sticking with $t = 0$. These functions are also given recursively by

$$\phi^{r+1}(s, t) = \sum_{k=1}^{n} (k+t)^{-s} \phi^r\left(s, \frac{1}{k+t}\right) \tag{1.3}$$

with $\phi^0(s, t) \equiv 1$.

The recursion (1.2) can be simulated on a computer, and this leads to estimates for the exponent of convergence of the series (1.1). The estimate $0.53128050 \pm 0.00000001$ for $\dim(E_2)$ was obtained on a 256K PC by first iterating a lower bound version of (1.3) with $s = 1.06256098$ some 20 times, until the stopping condition $\phi^{r+1}(s, t) > \phi^r(s, t)$ on $\frac{1}{3} \leqslant t \leqslant \frac{3}{4}$, or more accurately its discrete analog, was achieved. Once this happened, further iteration could only send the lower bound for ϕ^r on a slow exponential growth to infinity, which meant that this s is less than $2\dim(E_n)$. Then the exercise was repeated with $s = 1.06256102$ and an upper bound version of (1.3), and the stopping condition "$\phi^{r+1}(s, t) < \phi^r(s, t)$" on $\frac{1}{3} \leqslant t \leqslant \frac{3}{4}$ (discrete analog for the upper bound computed estimate) was again achieved after some 20 iterations.

There are many ways one might refine the procedure given here. First, use a bigger machine. The error in linear interpolation of a finite table for $\phi^r(s, t_i)$ is the main obstacle to accuracy, and this is inversely proportional to the square of the density of the table. Second, keep estimates also for $(d/dt)\,\phi^r(s, t)$ and $(d^2/dt^2)\,\phi^r(s, t)$. These can then be used to improve the interpolation of (1.3).

2. The Normalized Asymptotic Behavior of ϕ^r

For $s \neq D(n)$, the exponent of convergence of (1.4), the iterates $\phi^r(s, t)$ tend to zero or infinity. To better understand their behavior, we look at their normalized versions, and throw in an extra parameter.

For $0 \leqslant \theta \leqslant 1$ and $v \in V_n(r)$, let $\langle v, \theta \rangle := \langle v_1, v_2, ..., v_r + \theta \rangle = \langle v \rangle + \theta \langle v_1, v_2, ..., v_{r-1} \rangle$. With the obvious meaning of $[v, \theta]$, let

$$\begin{aligned} \phi_\theta^0(s, t) &= (1 + \theta t)^{-s}, \\ \phi_\theta^r(s, t) &= \sum_{v \in V_n(r)} \langle v, \theta \rangle^{-s} \phi_{[v,\theta]}^0(s, t). \end{aligned} \tag{2.1}$$

Then for $r \geqslant 0$,

$$\phi_\theta^{r+1}(s, t) = \sum_{k=1}^{n} (k + t)^{-s} \phi_\theta^r \left(s, \frac{1}{k + t} \right). \tag{2.2}$$

Now let $\psi_0^0(s, t) \equiv 1$, and for $0 < \theta \leqslant 1$, put

$$\psi_\theta^0(s, t) := \frac{\theta(s-1)}{1 - (1+\theta)^{1-s}} (1 + \theta t)^{-s} = \phi_\theta^0(s, t) \Big/ \int_0^1 \phi_\theta^0(s, u)\, du,$$

and

$$\psi_\theta^r(s, t) := \phi_\theta^r(s, t) \Big/ \int_0^1 \phi_\theta^r(s, u)\, du. \tag{2.3}$$

If we wish to make the dependence on n explicit, we write $\psi_\theta^r(s, t, n)$.

For $1 \leqslant k \leqslant n$, let

$$\begin{aligned} a_k^r(s, \theta) &= \sum_{v \in V_n(r)} \int_0^1 \langle u + v_1, v_2, ..., v_r, k + \theta \rangle^{-s}\, du, \\ (a_k^0(s, \theta) &= \int_0^1 (u + k + \theta)^{-s}\, du) \end{aligned} \tag{2.4}$$

and put

$$\gamma_k^r(s, \theta) = a_k^r(s, \theta) \Big/ \sum_{l=1}^{n} a_l^r(s, \theta).$$

These definitions are generalizations to $s > 1$ of those found in [6, Lemma 2] for $s = 2$. The following identity exhibits $\psi_\theta^{r+1}(s, t)$ as a convex combination of various $\psi_{w_i}^r(s, t)$, $0 \leqslant w_i \leqslant 1$, and permits us to etablish the existence of a limiting function $g(s, t) = \lim_{r \to \infty} \psi_\theta^r(s, t)$ (uniformly on $0 \leqslant \theta \leqslant 1$ and $0 \leqslant t \leqslant 1$, for each $s > 1$. In fact, the convergence is even uniform in s.)

The proof of this identity for $s = 2$ is found in [6, Lemma 2], and goes over to the case $s > 1$ without difficulty: For $r \geqslant 0$,

$$\psi_\theta^{r+1}(s, t) = \sum_{k=1}^{n} \gamma_k^r(s, \theta)\, \psi_{1/(k+\theta)}^r(s, t). \tag{2.5}$$

Next, following [6], we obtain a peakedness comparison: If $\theta_1 < \theta_2$ then

$$\psi^0_{\theta_1}(s, t) \prec \psi^0_{\theta_2}(s, t). \tag{2.6}$$

(That is, $\int_0^t (\psi^0_{\theta_2}(s, u) - \psi^0_{\theta_1}(s, u))\, du > 0$ for $0 < t < 1$, with equality at $t = 0$ and at $t = 1$.)

Proof. We need, equivalently, if $\theta = \theta_1$ and $\theta + h = \theta_2$,

$$\begin{aligned} &\int_0^t (1 + \theta u)^{-s}\, du \int_0^1 (1 + (\theta + h)u)^{-s}\, du \\ &\qquad < \int_0^t (1 + (\theta + h)u)^{-s}\, du \int_0^1 (1 + \theta u)^{-s}\, du. \end{aligned} \tag{2.7}$$

Doing the integrations and a little algebra yields the equivalent condition (with $\sigma = s - 1$)

$$\frac{1 - (1 + \theta t)^{-\sigma}}{1 - (1 + \theta)^{-\sigma}} < \frac{1 - (1 + (\theta + h)t)^{-\sigma}}{1 - (1 + \theta + h)^{-\sigma}}, \qquad \text{for} \quad 0 < t < 1. \tag{2.8}$$

At $t = 0$ or 1, of course, we have equality in (2.8). Taking logs gives an equivalent condition

$$\begin{aligned} &\log(1 - (1 + \theta + h)^{-\sigma}) - \log(1 - (1 + \theta)^{-\sigma}) \\ &\qquad < \log(1 - (1 + (\theta + h)t)^{-\sigma}) - \log(1 - (1 + \theta t)^{-\sigma}). \end{aligned} \tag{2.9}$$

Now (2.9) will hold provided that for $0 < t < 1$, $(d/d\theta) \log(1 - (1 + \theta)^{-\sigma}) < (d/d\theta) \log(1 - (1 + \theta t)^{-\sigma})$. So we need (after a little algebra)

$$\frac{(1 + \theta)^{-\sigma - 1}}{1 - (1 + \theta)^{-\sigma}} < \frac{t(1 + \theta t)^{-\sigma - 1}}{1 - (1 + \theta t)^{-\sigma}}.$$

Cross multiplying, and then multiplying on both sides by $(1 + \theta)^{\sigma + 1}(1 + \theta t)^{\sigma + 1}$, simplifies this to $(1 + \theta t)^{\sigma + 1} - (1 + \theta t) < t((1 + \theta)^{\sigma + 1} - (1 + \theta))$, and then to

$$(1 + \theta t)^{\sigma + 1} - t(1 + \theta)^{\sigma + 1} < 1 - t \qquad (0 < \theta < 1, 0 < t < 1). \tag{2.10}$$

In (2.10) we have equality for $t = 0$ and for $t = 1$. Since the right side is linear in t, (2.10) will hold provided $(d^2/dt^2)((1 + \theta t)^{\sigma + 1} - t(1 + \theta)^{\sigma + 1}) > 0$ for $0 < t < 1$. But this is $\theta^2 \sigma(\sigma + 1)(1 + \theta t)^{\sigma - 1} > 0$ since $\sigma > 0$, which proves (2.6).

Now (2.6) is a special case of a more general peakedness relation, which we shall prove by induction.

LEMMA 1. *For* $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ *and* $r \geqslant 0$,

$$\psi^r_{\theta_1}(s, t) \prec \psi^r_{\theta_2}(s, t) \qquad (r \text{ even})$$

$$\psi^r_{\theta_1}(s, t) \succ \psi^r_{\theta_2}(s, t) \qquad (r \text{ odd}).$$

The iductive step depends on another peakedness relation. We have

$$\gamma^r_k(s, \theta_1) \succ \gamma^r_k(s, \theta_2) \tag{2.11}$$

(as a sequence indexed by k) under the same hypotheses as Lemma 1 and regardless of the parity of r.

For finite sequences of positive numbers (b_k), (c_k), $1 \leqslant k \leqslant n$, summing to 1, the meaning here of $(b_k) \succ (c_k)$ is just that $\sum_{j=1}^k b_j > \sum_{j=1}^k c_j$ for $1 \leqslant k < n$.

The proof of (2.11) rests on the technical

LEMMA 2. *For* $\theta > 0$, $r \geqslant 0$ *and* $s > 1$,

$$\frac{d^2}{d\theta^2} \log a^r_k(s, \theta) > 0.$$

Lemma 2 will take quite a bit of calculation to establish. We begin with the case $r = 0$, where (suppressing r and s), $a_k(\theta) = \int_0^1 (k + \theta + u)^{-s} \, du$, so that $a_k(\theta) \, a''_k(\theta) - a'^2_k(\theta) = s(\int_0^1 \int_0^1 (s+1)(k + \theta + u)^{-s-2}(k + \theta + u)^{-s-1}(k + \theta + x)^{-s-1} \, du \, dx)$. This will be positive provided that the integrand (call it $f(u, x)$, say) satisfies

$$f(u, x) + f(x, u) > 0.$$

Temporarily setting $K = k + \theta$ to simplify things, we require

$$\begin{aligned}(s+1)((u+K)^{-s-2}(x+K)^{-s} + (u+K)^{-s}(x+K)^{-s-2}) \\ > 2s(u+K)^{-s-1}(x+K)^{-s-1}.\end{aligned}$$

Multiplying by $(u+K)^{s+2}(x+K)^{s+2}$ reduces this to requiring that

$$(s+1)((u+K)^2 + (x+K)^2) > 2s(u+K)(x+K),$$

which is clearly true since $x > 0$. Thus $a^{0''}_k(\theta) \, a^0_k(\theta) > (a^{0'}_k(\theta))^2$ for $\theta > 0$, which is equivalent to Lemma 2 for $r = 0$.

For $r \geqslant 1$, we have

$$a_k(\theta) = \sum_{v \in V_n(r)} \int_0^1 (\langle v_1, v_2, ..., v_r, k+\theta\rangle + u\langle v_2, ..., v_r, k+\theta\rangle)^{-s}\, du. \tag{2.12}$$

Now with $J = 1/(k+\theta)$, and letting $v_- = (v_2, ..., v_r)$, $\delta = s+1$, this gives

$$\begin{aligned} a_k(\theta) &= \sum_{v \in V_n(r)} \int_0^1 (\langle v, J\rangle + u\langle v_-, J\rangle)^{1-\delta}\, du \\ a_k'(\theta) &= \sum_{v \in V_n(r)} (1-\delta) \int_0^1 \langle u+v_1, ..., v_r\rangle (\langle v, J\rangle + u\langle v_-, J\rangle)^{-\delta}\, du \end{aligned} \tag{2.13}$$

and

$$a_k''(\theta) = \delta(\delta-1) \sum_{v \in V_n(r)} \int_0^1 \langle u+v_1, ..., v_r\rangle^2 (\langle v, J\rangle + u\langle v_-, J\rangle)^{-\delta-1}\, du.$$

Thus Lemma 2 is equivalent to the requirement that

$$\begin{aligned} &\delta(\delta-1) \sum_{v \in V_n(r)} \sum_{w \in V_n(r)} \int_0^1 \int_0^1 F(v, w, u, x)\, dx\, du \\ &\qquad - (\delta-1)^2 \sum_{v \in V_n(r)} \sum_{w \in V_n(r)} \int_0^1 \int_0^1 G(v, w, u, x)\, dx\, du > 0, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} F(v, w, u, x) = {} & (\langle v\rangle + u\langle v_-\rangle)^2 (\langle v, J\rangle + u\langle v_-, J\rangle)^{-\delta-1} \\ & \cdot (\langle w, J\rangle + x\langle w_-, J\rangle)^{-1-\delta}, \end{aligned}$$

and

$$\begin{aligned} G(v, w, u, x) = {} & (\langle v\rangle + u\langle v_-\rangle)(\langle w\rangle + x\langle w_-\rangle)(\langle v, J\rangle + u\langle v_-, J\rangle)^{-\delta} \\ & \cdot (\langle w, J\rangle + x\langle w_-, J\rangle)^{-\delta}. \end{aligned}$$

Now let

$$\begin{aligned} A(u) &= \langle v, J\rangle + u\langle v_-, J\rangle \\ B(u) &= \langle v\rangle + u\langle v_-\rangle \\ C(x) &= \langle w, J\rangle + x\langle w_-, J\rangle \end{aligned}$$

and

$$D(x) = \langle w\rangle + x\langle w_-\rangle,$$

for fixed $v, w \in V_n(r)$. In the double sums of (2.14), the terms for which $v = w$ contribute, to (2.14) a total of

$$\sum_{v \in V_n(r)} \left\{ (s+1) \int_0^1 \int_0^1 B^2(u)\, A^{-s-2}(u)\, A^{-s}(x) - s \int_0^1 \int_0^1 B(u)\, B(x)\, A^{-s-1}(u)\, A^{-s-1}(x) \right\}.$$

This is positive because $(s+1)(A^{-s-2}(u)\, A^{-s}(x)\, B^2(x) + A^{-s-2}(x)\, A^{-s}(u)$ $B^2(u)) > 2sA^{-s-1}(u)\, A^{-s-1}(x)\, B(u)\, B(x)$, since on multiplying by $A^{s+2}(u)\, A^{s+2}(x)$ this last inequality reduces to $(s+1)(A^2(u)\, B^2(x) +$ $A^2(x)\, B^2(u)) > 2sA(u)\, B(x)\, A(x)\, B(u)$, which is trivial.

Next we show that the contribution to (2.14) from each pair of terms (v, w) and (w, v) (where $v \neq w$) is also positive. That is, we claim

$$(s+1) \int_0^1 \int_0^1 (A^{-s}(u)\, C^{-s-2}(x)\, D^2(x) + A^{-s-2}(u)\, B^2(u)\, C^{-s}(x))\, du\, dx$$
$$> 2s \int_0^1 \int_0^1 (A^{-s-1}(u)\, C^{-s-1}(x)\, B(u)\, D(x))\, du\, dx. \tag{2.15}$$

The effect of the $+1$ in $(s+1)$ on the left is to increase the left side, so (2.14) will hold with "$>$," provided it holds with "$\geqslant$" when the $+1$ is deleted. So we just need (again using symmetry in $\int_0^1 \int_0^1$) to show that

$$\begin{aligned} &A^{-s}(u)\, C^{-s-2}(x)\, D^2(x) + A^{-s-2}(x)\, C^{-s}(u)\, B^2(x) \\ &\quad + A^{-s}(x)\, C^{-s-2}(u)\, D^2(u) + A^{-s-2}(u)\, C^{-s}(x)\, B^2(u) \\ &\geqslant 2(A^{-s-1}(u)\, C^{-s-1}(x)\, B(u)\, D(x) + A^{-s-1}(x)\, C^{-s-1}(u)\, B(x)\, D(u)). \end{aligned}$$

This holds because the first and fourth terms on the left dominate the first term on the right, while the second and third terms dominate the second term on the right. The proof of this is simple: multiply by $A^{s+2}(u)\, C^{s+2}(x)$ and simplify the first claim to $A^2(u)\, D^2(x) + C^2(x)\, B^2(u) - 2A(u)\, C(x)$ $B(u)\, D(x) \geqslant 0$ and then to $(A(u)\, D(x) - B(u)\, C(x))^2 \geqslant 0$. Then proceed in a similar manner with the other claim. This completes the proof of (2.15), and with it (2.14) and Lemma 2. Now from this lemma $(d^2/d\theta^2) \log a_k(\theta)$ > 0, so $(d/d\theta) \log a_k(\theta) < (d/d\theta) \log a_k(\theta + 1)$, and so for $h > 0$,

$$\log a_k(\theta + h) - \log a_k(\theta) < \log a_{k+1}(\theta + h) - \log a_{k+1}(\theta) \tag{2.16}$$

or, equivalently,

$$a_k(\theta + h)/a_k(\theta) < a_{k+1}(\theta + h)/a_{k+1}(\theta).$$

Thus if $c_k := a_k(\theta + h)/a_k(\theta)$, then $0 < c_1 < c_2 \cdots < c_n$.

Now consider the sequences $(a_k(\theta))$, $1 \leqslant k \leqslant n$, and $(a_k(\theta+h))$, $1 \leqslant k \leqslant n$. The latter sequence is just $(c_k a_k(\theta))$. We now quote a result from [6], where it appears as Lemma 5.

LEMMA 3. *Suppose* $0 < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ *and* $a_1, a_2, ..., a_n > 0$. *Then for all* k, $1 \leqslant k \leqslant n$,

$$\sum_{j=1}^{k} a_j \Big/ \sum_{j=1}^{n} a_j \geqslant \sum_{j=1}^{k} c_j a_j \Big/ \sum_{j=1}^{n} c_j a_j.$$

From this lemma, with $(a_k) = a_k(\theta)$ and c_k as above, it follows that $(\gamma_k^r(\theta)) \succ (\gamma_k^r(\theta+h))$ for $r \geqslant 0$, $s > 1$, and $0 \leqslant \theta + h \leqslant 1$. This proves (2.11).

For the remainder of the proof of Lemma 1, we refer the reader to the proof of Lemma 2 of [6].

Of course we already have (2.6), which is the case $r = 0$. For the inductive step the same argument as that in the last stage of the proof of Lemma 2 of [6] applies.

We next extend the notation of [6] to define $\Psi_\theta^r(s, t) := \int_{u=0}^{t} \psi_\theta^r(s, u)\, du$ and note that for even r, $\Psi_\theta^r(s, t)$ is strictly increasing in θ for each t, $0 < t < 1$, and decreasing in θ for odd r. Thus $\Gamma^r(s, \theta) := \int_0^1 \Psi_\theta^r(s, u)\, du$ is monotone (increasing for r even, decreasing for r odd) in θ, $0 \leqslant \theta \leqslant 1$.

Now from (2.5),

$$\Gamma^{r+1}(s, \theta) = \sum_{k=1}^{n} \gamma_k^r(s, \theta)\, \Gamma^r\left(s, \frac{1}{k+\theta}\right). \tag{2.17}$$

Thus (suppressing the dependence on s),

$$\begin{aligned}\Gamma^{r+1}(0) - \Gamma^{r+1}(1) &= \sum_{k=1}^{n} \gamma_k^r(1)\, \Gamma^r\left(\frac{1}{k+1}\right) - \gamma_k^r(0)\, \Gamma^r\left(\frac{1}{k}\right) \\ &= \sum_{k=1}^{n+1} \delta(k, r)\, \Gamma^r(1/k),\end{aligned} \tag{2.18}$$

say. Because of cancellation among terms $\delta(k, r) = \gamma_k^r(1) - \gamma_{k+1}^r(0)$ for $2 \leqslant k \leqslant n$, the sum of the positive $\delta(k, r)$ is less than 1. (The sum of all the $\delta(k, r)$, $1 \leqslant k \leqslant n$, is of course zero.) This cancellation ensures that

$$|\Gamma^{r+1}(0) - \Gamma^{r+1}(1)| < |\Gamma^r(0) - \Gamma^r(1)|,$$

which means that the functions $\psi_\theta^r(s, t)$, $0 \leqslant \theta \leqslant 1$, are being squeezed together as r increases. The existence of a limiting function, which is independent of θ, is a key ingredient in our analysis of Hausdorff dimension.

It will pay for us to look closely at the rate of convergence, keeping explicit track of the arithmetic in the case of $n = 2$.

LEMMA 4. *For all s, $1.05 \leqslant s \leqslant 2.05$, and all $n \geqslant 2$, there exists a function*

$$g(s, t, n) := \lim_{r \to \infty} \psi^r_\theta(s, t, n).$$

Moreover, uniformly in $1.05 \leqslant s \leqslant 2.05$, $r \geqslant 1, 0 \leqslant \theta \leqslant 1$, and $0 \leqslant t \leqslant 1$, $g(s, t, n) = \psi^r_\theta(s, t, n)(1 + O((\frac{14}{15})^r)$. In the case $n = 2$, the $(\frac{14}{15})$ in the error estimate can be sharpened to (0.865) for $1.06 \leqslant s \leqslant 1.07$.

Proof. Recall that

$$\gamma^r_2(0) = \frac{\sum_{v \in V_2(r)} \int_0^1 \langle u + v_1, v_2, ..., v_r, 2\rangle^{-s}\, du}{\sum_{v \in V_2(r)} \sum_{k=1}^2 \int_0^1 \langle u + v_1, v_2, ..., v_r, k\rangle^{-s}\, du},$$

and

$$\begin{aligned}\gamma^0_2(0) &= \frac{\int_0^1 (u+2)^{-s}\, du}{\int_0^1 ((u+1)^{-s} + (u+2))^{-s}\, du} \\ &= \frac{(1/(1-s))(u+2)^{1-s}|_0^1}{(1/(1-s))\{(u+2)^{1-s} + (u+1)^{1-s}|_0^1\}} \\ &= \frac{2^{1-s} - 3^{1-s}}{1^{1-s} - 3^{1-s}} \in (0.36, 0.37)\end{aligned}$$

for $1.06 \leqslant s \leqslant 1.07$, while

$$\gamma^0_1(1) = \frac{\int_0^1 (u+2)^{-s}\, du}{\int_0^1 ((u+2)^{-s} + (u+3)^{-s})\, du} = \frac{2^{1-s} - 3^{1-s}}{2^{1-s} - 4^{1-s}} \in (0.59, 0.60).$$

Now $\langle u + v_1, v_2, ..., v_r, k\rangle = k\langle u + v_1, v_2, ..., v_r\rangle + \langle u + v_1, v_2, ..., v_{r-1}\rangle$, and $\langle u + v_1, v_2, ..., v_{r-1}\rangle \geqslant \frac{1}{3}\langle u + v_1, v_2, ..., v_r\rangle$, so $\langle u + v_1, v_2, ..., v_r, 1\rangle \geqslant \frac{4}{7}\langle u + v_1, v_2, ..., v_r, 2\rangle$. Thus $\sum_{v \in V_2(r)} \sum_{k=1}^2 \int_0^1 \langle u + v_1, v_2, ..., v_r, k\rangle^{-s}\, du \leqslant 2.82 \sum_{v \in V_2(r)} \int_0^1 \langle u + v_1, v_2, ..., v_r, 2\rangle^{-s}\, du$, for $1.06 \leqslant s \leqslant 1.07$, so that $\gamma_2(0) \geqslant 0.354$. Since clearly $\gamma^r_1(1) > \gamma^r_2(0)$, this shows that

$$|\Gamma^{r+1}(0) - \Gamma^{r+1}(1)| \leqslant 0.646\, |\Gamma^r(0) - \Gamma^r(1)|. \tag{2.19}$$

Originally, $\Gamma^0(0) - \Gamma^0(1) = ((s-1)/(1-2^{1-s})) \int_0^1 \int_0^t (1+u)^{-s}\, du - \frac{1}{2} = (1/(1-2^{1-s})) \int_0^1 (1-(1+t)^{1-s})\, dt - \frac{1}{2} = 1/(1-2^{1-s}) - \frac{1}{2} - (2^{2-s}-1)/(2-s)(1-2^{1-s}) < \frac{1}{16}$ for $1.06 \leqslant s \leqslant 1.07$. Therefore for $1.06 \leqslant s \leqslant 1.07$,

$$|\Gamma^r(0) - \Gamma^r(1)| \leqslant \tfrac{1}{16}(0.646)^r. \tag{2.20}$$

Now if $F(t) = \Psi^r_{\theta_1}(s, t) - \Psi^r_{\theta_2}(s, t)$, $0 \leqslant \theta_i \leqslant 1$, then $F(t)$ is a linear combination of functions of the form $((1+\theta t)^{1-s} - 1)/((1+\theta)^{1-s} - 1)$, with all coefficients in $[-1, 1]$, the sum of the positive coefficients is 1, and the sum of the negative coefficients is -1. Note that $F(0) = F(1) = 0$.

The largest possible value for $|F''(t)|$ occurs with $t=0$, $\theta_1=1$, and $\theta_2=0$, with coefficients $C_1=1$ and $C_2=-1$, in which case $|F''(t)|=s(s-1)/(1-2^{1-s})$. For $1.06 \leqslant s \leqslant 1.07$, this is $\leqslant 1.6$. Since one of $\Psi^r_{\theta_i}(s, t)$ is more peaked then the other, $F(t)$ has constant sign on $(0, 1)$.

If $F(c)$ is the largest value of $F(t)$ on $(0, 1)$, then $p(t) := F(c) - 0.8(t-c)^2$ must be $\leqslant 0$ at 0 and 1. Thus $\int_0^1 F(t)\, dt \geqslant \frac{2}{3}\sqrt{5}(F(c))^{3/2}$, and so

$$F(c) \leqslant (0.206)(0.748)^r, \tag{2.21}$$

from (2.20) and the foregoing.

Now in view of the bound on $|F''(t)|$, if $|F'(c_1)| = D$ then $|F'(c_1+h)| \geqslant D - 1.6h$. Thus there is an interval of length at least $\frac{5}{8}D$ on which $|F(t)| \geqslant |F(c_1)| + |t-c_1|\, D - \frac{4}{5}|t-c_1|^2$. The integral of $F(t)$ over this interval is at least $\frac{25}{96}D^3$, so $D \leqslant (\frac{96}{25})^{1/3} \cdot (\frac{1}{16})^{1/3}(0.646)^{r/3}$. That is, $|F'(t)| \leqslant (0.63)(0.865)^r$, so that

$$|\psi'^r_{\theta_1}(s, t, 2) - \psi'^r_{\theta_2}(s, t, 2)| \leqslant (0.63)(0.865)^r, \tag{2.22}$$

uniformly in $r \geqslant 0$, $0 \leqslant t \leqslant 1$, $1.06 \leqslant s \leqslant 1.07$, and $0 \leqslant \theta_i \leqslant 1$.

A similar analysis shows that there exists an $\varepsilon > 0$ and $C > 0$ such that for all $n \geqslant 2$, all s, $1.05 \leqslant s \leqslant 2.05$, all $r \geqslant 0$, all t, $0 \leqslant t \leqslant 1$, and all θ_1, θ_2 also in $[0, 1]$,

$$|\psi'^r_{\theta_1}(s, t, n) - \psi'^r_{\theta_2}(s, t, n)| \leqslant C(1-\varepsilon)^r. \tag{2.23}$$

In fact, the cancellation in the passage from Γ^r to Γ^{r+1} involves $\sum_{j=2}^n (\gamma^r_{j-1}(1) - \gamma^r_j(0))$, and since from their definitions $\gamma^r_{j-1}(1) > \gamma^r_j(0)$, the total cancellation comes to $\sum_{j=2}^n \gamma^r_j(0) = 1 - \gamma^r_1(0)$. For $1 < s \leqslant 2.05$ and $n \geqslant 2$, it is easily seen from the definition of $\gamma^r_1(0)$ that

$$\gamma^r_1(0) < \left(\sum_{j=1}^n j^{-s}\right)^{-1} < \left(\sum_{j=1}^2 j^{-2.05}\right)^{-1} < 0.81.$$

By the same sort of argument as that in the proof of (2.22), we can thus take $C=2$ and $\varepsilon = 1-(0.81)^{1/3}$, or for simplicity, say $\varepsilon = \frac{1}{15}$, in (2.23). This proves Lemma 4.

3. Properties of $g(s, t, n)$

As in [6], there exists, for each s, $1.05 \leqslant s \leqslant 2.05$, a probability distribution function $\beta_{s,n}(\theta)$ such that

$$g(s, t, n) = \int_{\theta=0}^1 w(s, \theta)(1+\theta t)^{-s}\, d\beta_{s,n}(\theta), \tag{3.1}$$

where $w(s, \theta) = (s-1)\,\theta/(1-(1+\theta)^{1-s})$ so that $w(s, \theta)(1+\theta t)^{-s} = \psi^0_\theta(s, t)$.

From this, it follows that

$$(-1)^j \frac{d^j}{dt^j} g(s,t,n) > 0 \qquad \text{for all} \quad j \geqslant 0 \text{ and all } t, 0 \leqslant t \leqslant 1, \tag{3.2}$$

whenever $1.05 \leqslant s \leqslant 2.05$ and $n \geqslant 2$. We require the following information about $g(s, t, n)$.

THEOREM 1. *For all $n \geqslant 2$, for all s, $1.05 \leqslant s \leqslant 2.05$, and for all $t, 0 \leqslant t \leqslant 1$,*

(i) $|g(s,t,n) - \psi_\theta^r(s,t,n)| \leqslant 2(0.94)^r$ *for* $0 \leqslant \theta \leqslant 1$ *while* $|g(s,t,2) - \psi_\theta^r(s,t,2)| \leqslant (0.63)(0.865)^r$ *under the same assumptions.*

(ii) *If* $\lambda(s,n) := \sum_{k=1}^n \int_0^1 (k+u)^{-s} g(s, 1/(k+u), n)\, du$, *then* $\lim_{r\to\infty}(1/r)\log \phi_0^r(s,0,n) = \log \lambda(s,n)$.

(iii) $\lambda(s,n)\, g(s,t,n) = \sum_{k=1}^n (k+t)^{-s} g(s, 1/(k+t), n)$

(iv) *For each s there exists a constant $C(s,n)$ such that* $|C(s,n)\, \phi_0^r(s,t,n) - (\lambda(s,n))^r g(s,t,n)| \leqslant (0.94)^r(\lambda(s,n))^r$

(v) *For $n \geqslant 2$ there exists a $D = D(n)$ such that $\lambda(D(n), n) = 1$, and this D is twice the Hausdorff dimension of E_n.*

Before proving Theorem 1, we note how it can serve to determine the Hausdorff dimension. In [7], it is shown that $\log \lambda(s,n)$ is convex and decreasing in s, and that $\log \lambda(s+h,n) - \log \lambda(s,n) \leqslant -h \log((1+\sqrt{5})/2)$. From Theorem 1(ii), $\lambda(s,n)$ as defined here is the same as the $\lambda(s,n)$ of [7], where it was defined as $\lim_{r\to\infty}(1/r)\log \phi_0^r(s,0,n)$.

Since $\lambda(s,n)$ is decreasing in s, if $\lambda(s_1,n) > 1 > \lambda(s_2,n)$ then $s_1 < D(n) < s_2$. From Lemma 1 of [7], $\log(\phi_0^r(s,0,n)(\lambda(s,n))^{-r})$ is bounded as $r \to \infty$, uniformly in $1 \leqslant s \leqslant 2$. (The proof given there serves equally well for $s \leqslant 2.05$, but the explicit bound of $\log 4$ would have to be relaxed to $2\log(2.05)$.) This gives

$$\text{If } \lim_{r\to\infty} \phi_0^r(s_1,t,n) = \infty \text{ and } \lim_{r\to\infty} \phi_0^r(s_2,t,n) = 0,$$

$$\text{then } \lambda_1(s_1,n) > 1 > \lambda(s_2,n) \text{ and } s_1 < D(n) < s_2. \tag{3.3}$$

Remark. The computational work will involve estimating $\phi_0^r(s,t,n)$, by way of (2.2), so as to find s_1 and s_2, with $|s_1 - s_2|$ small, as in (3.3).

We now turn to the proof of Theorem 1. The first item is Lemma 4. Now the functions $\psi_\theta^0(s,t)$ are decreasing in t, yet for $0 \leqslant \theta \leqslant 1$, $1.05 \leqslant s \leqslant 2.05$, $\psi_\theta^0(s,1) \geqslant \frac{1}{5}\psi_\theta^0(s,0)$. Thus the same holds for the $\psi_\theta^0(s,t,n)$, which are positive linear combinations of various $\psi_{\theta_i}^0(s,t)$, and so also for their common limit. That is, for $0 \leqslant t \leqslant 1$ and $1.05 \leqslant s \leqslant 2.05$,

$$\tfrac{1}{5} g(s,0,n) \leqslant g(s,t,n) \leqslant g(s,0,n). \tag{3.4}$$

From Lemma 4 and the above, it follows that $\phi_0^{r+1}(s, t, n)(\phi_0^r(s, t, n))^{-1} = (1 + O(\frac{14}{15})^r)\, \phi_0^{r+1}(s, 0, n)(\phi_0^r(s, 0, n))^{-1}$, uniformly in $1.05 \leqslant s \leqslant 2.05$, $r \geqslant 1$, $n \geqslant 2$, and $0 \leqslant t \leqslant 1$. Let $K(r, s, n) := \phi_0^{r+1}(s, 0, n)(\phi_0^r(s, 0, n))^{-1}$.

On the other hand, from (1.3) we have

$$\begin{aligned}
&\phi_0^{r+1}(s, t, n)(\phi_0^r(s, t, n))^{-1} \\
&\quad = \sum_{k=1}^{n} (k+t)^{-s} \phi_0^r(s, (k+1)^{-1}, n)(\phi_0^r(s, t, n))^{-1} \\
&\quad = \sum_{k=1}^{n} (k+t)^{-s} \psi_0^r(s, (k+t)^{-1}, n)(\psi_0^r(s, t, n))^{-1} \qquad \text{by (2.3).}
\end{aligned}$$

By Lemma 4 and (3.4), though, this is $(1 + O(\frac{14}{15})^r) \sum_{k=1}^{n} (k+t)^{-s} g(s, (k+t)^{-1}, n)(g(s, t, n))^{-1}$. That is,

$$K(r, s, n) = \left(1 + O\left(\frac{14}{15}\right)\right)^r \sum_{k=1}^{n} (k+t)^{-s} g(s, (k+t)^{-1}, n)(g(s, t, n))^{-1}. \tag{3.5}$$

Since the left side of (3.5) is independent of t while the right side, apart from the factor $(1 + O(\frac{14}{15})^r)$, is independent of r, it follows that $K(s, n) := \lim_{r \to \infty} K(r, s, n)$ exists, and that $K(r, s, n) = (1 + O(\frac{14}{15})^r)\, K(s, n)$. From (3.5), $K(s, n) = \sum_{k=1}^{n} (k+t)^{-s} g(s, (k+t)^{-1}, n)(g(s, t, n))^{-1}$, on sending r to infinity on both sides. Equivalently

$$K(s, n)\, g(s, t, n) = \sum_{k=1}^{n} (k+t)^{-s} g(s, (k+t)^{-1}, n). \tag{3.6}$$

Integrating with respect to t over $[0, 1]$ gives

$$K(s, n) = \sum_{k=1}^{n} \int_0^1 (k+u)^{-s} g(s, (k+u)^{-1}, n)\, du = \lambda(s, n). \tag{3.7}$$

Now (iii) of Theorem 1 follows from (3.5) together with (3.7).

If we let L be the positive linear transformation

$$Lf(t) := \sum_{k=1}^{n} (k+t)^{-s} f\left(\frac{1}{k+t}\right), \tag{3.8}$$

then $Lg(s, t, n) = \lambda(s, n)\, g(s, t, n)$, while $L\phi_0^r(s, t, n) = \phi_0^{r+1}(s, t, n)$.

Since L is a positive operator, and since $g(s, t, n) \ll \phi_\theta^0(s, t, n) \ll g(s, t, n)$, it follows that $\lambda^r g = L^r g \ll \phi_0^r \ll L^r g = \lambda^r g$, uniformly on $0 \leqslant t \leqslant 1$ as $r \to \infty$. From this it follows that $(1/r) \log \sum_{v \in V_n(r)} \langle v \rangle^{-s} = (1/r) \log \phi_0^r(s, 0, n) \to$

$\log \lambda(s, n)$ as $r \to \infty$. This proves (ii). Now (iv) holds because if we put $C(r, s, n) := (\lambda(s, n))^r g(s, 0, n)/\phi_0^r(s, 0, n)$, then from (i) of this theorem,

$$C(r, s, n)\,\phi_0^r(s, t, n) = (\lambda(s, n))^r g(s, t, n)(1 + O(0.94)^r) \tag{3.9}$$

uniformly on $0 \leqslant t \leqslant 1$. But then

$$L\{C(r, s, n)\,\phi_0^r(s, t, n)\} = L\{(\lambda(s, n))^r g(s, t, n)\} + O(0.94^r \lambda(s, n))^r, \tag{3.10}$$

so $C(r, s, n)\,\phi_0^{r+1}(s, t, n) = (\lambda(s, n))^{r+1} g(s, t, n)(1 + O(0.94)^r)$. Since also $C(r+1, s, n)\,\phi_0^{r+1}(s, t, n) = (\lambda(s, n))^{r+1} g(s, t, n)(1 + O(0.94)^r)$, we conclude that

$$C(r+1, s, n) = C(r, s, n)(1 + O(0.94)^r). \tag{3.11}$$

Thus if we put $C(s, n) := \lim_{r \to \infty} C(r, s, n)$ then the limit exists and is positive (and is reached fairly quickly.) Now (iv) follows from (3.4), (3.9), and (3.11), and again (i) of this theorem. For $n = 2$, we can do a little better. We have, for each s, $1.05 \leqslant s \leqslant 1.07$ and $0 \leqslant t \leqslant 1$,

$$C(r, s, 2)\,\phi_0^r(s, t) = (\lambda(s))^r g(s, t)(1 + 2Mh^r), \tag{3.12}$$

where $-1 \leqslant M \leqslant 1$ and $h = (0.865)$. This is because the relative error is zero at $t = 0$ (by definition), and the estimate $|\psi_0^r(s, t) - g(s, t)| \leqslant (0.63)h^r$ implies $|\psi_0^r(s, t)/g(s, t) - 1| \leqslant 2h^r$, since $\frac{1}{3} \leqslant g(s, t) \leqslant 3$ for $0 \leqslant t \leqslant 1$ and $1.05 \leqslant s \leqslant 1.07$. (All the $\psi_\theta^0(s, t)$ have this property, and $g(s, t)$ is a limit of convex combinations of them.) But then for $0 \leqslant t \leqslant 1$, $\phi_0^r(s, t)/(g(s, t)(\lambda(s))^r)$ is also constant (as a function of t) to within a factor of $1^{\pm} 2h^r$, as claimed. Refining the argument that gave (3.11) now gives

$$C(r+1, s, 2) = C(r, s, 2)(1 + 2M_1 h^{r+1})/(1 + 2M_2 h^r), \tag{3.13}$$

with $|M_1| \leqslant 1$ and $|M_2| \leqslant 1$. Now again since $g(s, t)$ is a convex combination of $\psi_\theta^0(s, t)$'s, and since it is easily seen that $\frac{1}{3} \leqslant \psi_{\theta_1}^0(s, t)/\psi_{\theta_2}^0(s, t) \leqslant 3$ for all (s, t) in our range and $\theta_1, \theta_2 \in [0, 1]$, it follows that in this range, and for $n = 2$,

$$\begin{aligned} &\tfrac{1}{3} \leqslant g(s, t) \leqslant 3, \\ &\tfrac{1}{3} \leqslant \psi_\theta^0(s, t)/g(s, t) \leqslant 3. \end{aligned} \tag{3.14}$$

Since L^r is a positive operator, it therefore follows that $\frac{1}{3} \leqslant C(r, s) \leqslant 3$, so that also $\frac{1}{3} \leqslant C(s) \leqslant 3$.

Most of (v) is contained in the literature. Cusick [2] proved that the abscissa of convergence of $\sum_{r=0}^{\infty} \phi_0^r(s, t, n)$ is twice the Hausdorff dimension of E_n, and that therefore if $\lambda(s_1, n) > 1 > \lambda(s_2, n)$ then $s_1/2 \leqslant \dim(E_n) \leqslant$

$s_2/2$. Since $\lambda(s, n)$ is strictly decreasing (see [7] (2.11), but this fact can hardly have been unknown), the only snag is the possibility of a jump discontinuity at $D(n)$. But $\log \lambda(s, n)$ is convex ([7], in the proof of (2.18)), so there can be no such discontinuity in the interior of the interval on which $\lambda(s, n)$ is defined. This completes the proof of (v), and of Theorem 1.

Inasmuch as $\dim(E_2)$ is known to be in [0.5312, 0.5314], it is clear that $\lambda(1.05,2)>1$, so that $\lambda(1.05, n)>1$ for $n \geqslant 2$. Also $\dim(E_n)<1$, [8], or see [7] so $\lambda(2, n)<1$. Thus $1.05<D(n)<2$ for $n \geqslant 2$.

4. Application to the Determination of $\dim(E_2)$

As we have seen, if $s_1<s_2$, $\lim_{r \to \infty} \phi_0^r(s_1, t, n)=\infty$ and if $\lim_{r \to \infty} \phi_0^r(s_2, t, n)=0$, then $s_1/2<\dim(E_n)<s_2/2$. Now L is a positive linear operator. Thus if $\phi_0^{r+1}(s_1, t, n)>\phi_0^r(s_1, t, n)$ for $0 \leqslant t \leqslant 1$ for some r, then $\lim_{r \to \infty} \phi_0^r(s_1, t, n)=\infty$. Likewise if for some r, $\phi_0^{r+1}(s_2, t, n)<\phi_0^r(s_2, t, n)$ for $0 \leqslant t \leqslant 1$, then $\lim_{r \to \infty} \phi_0^r(s_2, t, n)=0$.

For small r and for $s_1=D(n)-\varepsilon$, $s_2=D(n)+\varepsilon$, the above conditions will not be met, because of the alternating peakedness of $\psi_0^r(s, t, n)$ with increasing r. But $\phi^r(s, t, n)=(C(s, n))^{-1}(\lambda(s, n))^r g(s, t, n)(1+O(0.94)^r)$ from Theorem 1, (iv). Also, $\lambda(s_1, n)>1+\varepsilon \log((1+\sqrt{5})/2)$ while $\lambda(s_2, n)<1-\varepsilon \log((1+\sqrt{5})/2)$. Thus there exists a constant A such that for $r \geqslant A \log(1/\varepsilon)$, $\phi_0^{r+1}(D(n)-\varepsilon, t, n)>\phi_0^r(D(n)-\varepsilon, t, n)$ and $\phi_0^{r+1}(D(n)+\varepsilon, t, n)<\phi_0^r(D(n)+\varepsilon, t, n)$, for all t, $0 \leqslant t \leqslant 1$. That is,

> the number of iterations of L required to determine whether or not $s>D(n)$ is $O(\log(1/|s-D(n)|))$. (4.1)

In practice, exact calculation of $\phi_0^r(s, t, n)$ is not feasible since we should be forced either to proceed formally, by way of the identity (2.1), and suffer the resulting combinatorial explosion, or to keep values for $\phi_0^r(s, t, n)$ on some dense subset of [0, 1]. Instead, we shall keep upper bounds, when we think $s>D(n)$, and lower bounds when we think $s<D(n)$. A binary chop can be used to close in on $D(n)$.

Let N be a positive integer (chosen as large as available computer memory will permit), and let $T=T_N$ be the (positive linear) operator which replaces a function $f\colon [0, 1] \to \mathbf{R}$ with its linear interpolation based on the values of f at $\frac{1}{3}+\frac{5}{12}(j/N)$ ($\frac{1}{3} \leqslant t \leqslant \frac{3}{4}$), $0 \leqslant j \leqslant N$.

Let $L^+f(t)=TLFf(t)$. Now if $f(t) \geqslant \phi_0^r(t)$ then $Tf \geqslant T\phi_0^r$, so that $LTf \geqslant LT\phi_0^r$, and finally $TLTf \geqslant TLT\phi_0^r$. But then we have $T\phi_0^r \geqslant \phi_0^r$ (since ϕ_0^r is convex), so that $LT\phi_0^r \geqslant L\phi_0^r$ (since L is positive), and so $TLT\phi_0^r \geqslant TL\phi_0^r$ (since T is positive). Assembling the inequalities, we have

$TLTf \geqslant TL\phi_0^r \geqslant L\phi_0^r = \phi_0^{r+1}$, and if we start with $F^0 := 1$ and put $F^{r+1} = TLTF^r$, then we get

$$F^{r+1} \geqslant \phi_0^{r+1}. \tag{4.2}$$

Apart from possible round-off errors, F^r is directly machine computable. If the evaluations of $(k+t)^{-s}$, the linear interpolations, and additions are performed in double precision arithmetic and are accurate to, say, 1 part in 10^{13}, then $(1+10^{-12})^r$. (Machine evaluation of $F^r) \geqslant$ true F^r. For the values of r we will be using ($r \leqslant 100$), errors of $\leqslant 3 \times 10^{-10}$ in the machine evaluation of $F^r(t)$ are negligible.

Next we need a lower bound for ϕ_0^r. Let $Qf(t) := (1 - Cu(1-u))\, Tf(t)$, where $u = u(t) := N(t - t^*), t^*$ is the largest of the interpolation points $\frac{1}{3} + j/N$ for which $t^* \leqslant t$, and where C is between 0 and 1, reserving the choice for later. The idea is to choose it small, yet large enough that for all θ, $\frac{1}{3} \leqslant \theta \leqslant \frac{3}{4}$,

$$Q\phi_\theta^0(t) \leqslant \phi_\theta^0(t) \qquad \text{on} \quad \tfrac{1}{3} \leqslant t \leqslant \tfrac{3}{4}. \tag{4.3}$$

Q is again a positive linear operator (and like T, idempotent). Then let $L^- := QLQ$.

If (4.3) is satisfied, and if we put $f_0 \equiv 1$, $f^r := (QLQ)^r f^0 = (L^-)^r f^0$, then $\phi_0^0 \geqslant f^0$. If $\phi_0^r = f^r$ then $\phi_0^{r+1} = L\phi_0^r \geqslant QL\phi_0^r$ (by (4.3) since $L\phi_0^r$ is a positive linear combination of functions ϕ_θ^0), and $QL\phi_0^r \geqslant QLQ\phi_0^r$ since QL is positive and $\phi_0^r \geqslant Q\phi_0^r$, this last again by (4.3). Thus by induction, we get

$$\phi_0^r(t) \geqslant f^r(t), \qquad \tfrac{1}{3} \leqslant t \leqslant \tfrac{3}{4}. \tag{4.4}$$

Again multiplication by $(1 - 10^{-12})^r$ drowns out any round-off errors committed in the machine calculation of $f^r(t)$. So let

$$\begin{aligned} a_r(s, j) &= (1 - 10^{-12})^r \text{ Machine } f^r\left(\frac{1}{3} + \frac{5}{12}\frac{j}{2500}\right) \\ b_r(s, j) &= (1 + 10^{-12})^r \text{ Machine } F^r\left(\frac{1}{3} + \frac{5}{12}\frac{j}{2500}\right). \end{aligned} \tag{4.5}$$

If for some $r > 1$, $a_{r+1}(j) > a_r(j)$ for $0 \leqslant j \leqslant 2500$, then $s < 2\dim(E_2)$, while if for some $r > 1$, $b_{r+1}(j) < b_r(j)$ for $0 \leqslant j \leqslant 2500$, then $s > 2\dim(E_2)$.

So, how small can we take C? Well, when $1.05 \leqslant s \leqslant 1.07$, we have

$$\begin{aligned} \phi_\theta^0(s, t) &= (1 + \theta t)^{-s} \\ \frac{d}{dt}\phi_\theta^0(s, t) &= -\theta s(1 + \theta t)^{-s-1} \end{aligned}$$

and

$$\frac{d^2}{dt^2}\phi_\theta^0(s,t)=\theta^2 s(s+1)(1+\theta t)^{-s-2}.$$

Thus $(d^2/dt^2)\,\phi_\theta^0(s,t)/\phi_\theta^0(s,t)=\theta^2 s(s+1)(1+\theta t)^{-2}$. Now $\theta/(1+\theta t)$ is increasing in θ for $\frac{1}{3}\leqslant\theta\leqslant\frac{3}{4}$ and fixed t, $\frac{1}{3}\leqslant t\leqslant\frac{3}{4}$, so (d^2/dt^2) $\phi_\theta^0(s,t)/\phi_\theta^0(s,t)\leqslant\frac{9}{16}(1.07)(2.07)(\frac{5}{4})^{-2}=0.797364$. Now $Q\phi_\theta^0(s,t)$ agrees with $\phi_\theta^0(s,t)$ at $\frac{1}{3}+\frac{5}{12}(j/N)$, $0\leqslant j\leqslant N$.

It is easily seen that if two functions f_1 and f_2 agree at a and b, $a<b$, and if $f_1''>f_2''$ for $a<t<b$, then $f_1<f_2$ for $a<t<b$. So we need only show that

$$\frac{d^2}{dt^2}Q\phi_\theta^0(s,t)>\frac{d^2}{dt^2}\phi_\theta^0(s,t)\qquad\text{on the intervals}\left(\frac{1}{3}+\frac{5}{12}\frac{j}{N},\frac{1}{3}+\frac{5}{12}\frac{j+1}{N}\right). \tag{4.6}$$

Now $(d^2/dt^2)\,\phi_\theta^0(s,t)\leqslant\frac{9}{16}(1.07)(2.07)(\frac{5}{4})^{-2}<(0.8)$ on $[\frac{1}{3}\leqslant t\leqslant\frac{3}{4}]\times[\frac{1}{3}\leqslant\theta\leqslant\frac{3}{4}]$. But $(d^2/dt^2)\,Q\phi_\theta^0(s,t)=(d^2/dt^2)(1-Cu+Cu^2)(u\phi_\theta^0(\frac{1}{3}+5j/12N)+(1-u)\,\phi_\theta^0(\frac{1}{3}+\frac{5}{12}((j+1)/N))$, where here $u=(\frac{12}{5})N(t-\frac{1}{3}-\frac{5}{12}(j/N))$. Thus $d^2/dt^2=(\frac{12}{5}N)^2(d^2/du^2)$, and if we put $A=\phi_\theta^0(a)$, $B=\phi_\theta^0(b)$, this gives (after a little algebra)

$$\begin{aligned}\frac{d^2}{dt^2}Q\phi_\theta^0(s,t)&=\frac{144N^2}{25}\frac{d^2}{du^2}(B+(A-B-BC)u\\&\quad+(2BC-AC)u^2+(AC-BC)u^3)\\&=\frac{72N^2}{50}C(B+(B-A)(1-3u)).\end{aligned}\tag{4.7}$$

Now $\phi_\theta^0(s,t)$ is decreasing, so $B<A$ and the minimum value of $B+(B-A)(1-3u)$ on $0\leqslant u\leqslant 1$ occurs when $u=0$. Since $((\phi_\theta^0/dt)/\phi_\theta^0)=-\theta s(1+\theta t)^{-1}>-\frac{1}{3}$,

$$\begin{aligned}&\phi_\theta^0\left(s,\frac{1}{3}+\frac{5}{12}\left(\frac{j+1}{N}\right)\right)\Big/\phi_\theta^0\left(s,\frac{1}{3}+\frac{5}{12}\frac{j}{N}\right)\\&\quad>\exp\left(-\frac{1}{3}\cdot\frac{5}{12N}\right)>1-1/(7N),\end{aligned}\tag{4.8}$$

and so $B+(B-A)(1-3u)>((7N-2)/(7N-1))B$. Hence, for $1.05\leqslant s\leqslant 1.07$, and $\frac{1}{3}\leqslant 0$, $t\leqslant\frac{3}{4}$,

$$\frac{d^2}{dt^2}Q\phi_\theta^0(s,t)>\frac{7N-2}{7N-1}\frac{3C}{5}\cdot\frac{72N^2}{25}>5.7CN^2. \tag{4.9}$$

Thus we need only choose C so that $5.7CN^2 > 0.8$. With $N = 2500$ (about the best a 256K PC can handle), we can take $C = 1.2 \times 10^{-8}$. With 20 iterations of L^+ and L^-, I got

$$0.53128049 < \dim E_2 < 0.53128051. \tag{4.10}$$

It takes about a day to machine evaluate $(L^{\pm})^{25}\phi_0^0 = F^{25}$ or f^{25}, but time is hardly so expensive on a PC that this is a problem (in interpreted basic, and using a double precision series expansion subroutine to accurately carry out the evaluations of t^s).

Final remark. An adaptive procedure which estimates $(d^2/dt^2)\phi_0^r(s, t)$ from $f^r(\frac{1}{3} + (j-1)/N) - 2f^2(\frac{1}{3} + j/N) + f^r(\frac{1}{3} + (j+1)/N)$, and uses this to improve the interpolations used for $f^r(1/(k+t))$, should yield several more digits, albeit at the expense of rigor.

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